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1999 J. Phys. A: Math. Gen. 32 7901

(http://iopscience.iop.org/0305-4470/32/45/307)

J. Phys. A: Math. Gen. **32** (1999) 7901–7906. Printed in the UK PII: S0305-4470(99)04211-0

Bäcklund transformation and Lax pair for an extended integrable differential-difference system

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Received 10 May 1999, in final form 26 August 1999

Abstract. An integrable differential-difference system is proposed which is an extended form of the integrable differential-difference equation found by Hu and Wu (1998 *Phys. Lett.* A **246** 523–9). It is shown that this extended differential-difference system is integrable in the sense of having a Bäcklund transformation and a Lax pair.

Recently, the following new differential-difference equation was found [1]:

$$\begin{aligned}u_{tt}(n+1) + u_{tt}(n) + u_{tt}(n-1) - 3u(n)(u_t(n+1) + u_t(n-1))\\ +3u(n+1)u_t(n+1) + 3u(n-1)u_t(n-1) - \tfrac{1}{4}u(n+1) - \tfrac{1}{4}u(n-1)\\ +\tfrac{1}{2}u(n) + [u(n+1) - 2u(n) + u(n-1)][(u(n+1) - u(n-1))^2\\ -(u(n+1) - u(n))(u(n) - u(n-1))] = 0.\end{aligned} \tag{1}$$

This equation may be transformed into a pair of bilinear equations

$$\left(D_z e^{\frac{1}{2}D_n} - D_t^2 e^{\frac{1}{2}D_n}\right) f(n)\cdot f(n) = 0 \tag{2}$$

$$\left(D_t^3 e^{\frac{1}{2}D_n} + 3D_tD_z e^{\frac{1}{2}D_n} - D_t e^{\frac{1}{2}D_n}\right) f(n)\cdot f(n) = 0 \tag{3}$$

by the dependent variable transformation $u(n) = (\ln f(n))_t$, where z is an auxiliary variable and the bilinear operators are defined as follows [2–5]:

$$D_z^m D_t^k\, a\cdot b \equiv \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial z'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^k a(z,t)b(z',t')|_{z'=z,t'=t}$$

$$\exp(\delta D_n)a(n)\cdot b(n) \equiv \exp\left[\delta\left(\frac{\partial}{\partial n} - \frac{\partial}{\partial n'}\right)\right]a(n)b(n')|_{n'=n} = a(n+\delta)b(n-\delta).$$

It has been shown that equation (1) is integrable in the sense of having a Bäcklund transformation and a Lax pair [1, 6]

In this brief paper, the problem of integrable extensions of equation (1) will be considered. We will propose an integrable extension of equation (1). Our starting point is to generalize equations (2) and (3) first. After some tests and guesswork, we now propose the following generalized form of (2) and (3):

$$\left(D_z e^{\frac{1}{2}D_n} - D_t^2 e^{\frac{1}{2}D_n} + \tfrac{1}{2}\alpha\left(e^{\frac{3}{2}D_n} - e^{\frac{1}{2}D_n}\right)\right) f(n)\cdot f(n) = 0 \tag{4}$$

$$\left(D_t^3 e^{\frac{1}{2}D_n} + 3D_tD_z e^{\frac{1}{2}D_n} - D_t e^{\frac{1}{2}D_n} - \tfrac{3}{2}\alpha D_t e^{\frac{3}{2}D_n}\right) f(n)\cdot f(n) = 0 \tag{5}$$

where α is an arbitrary constant. By using bilinear operator identities

$$\sinh(\tfrac{1}{2}D_n)\big[\big(D_t^3 e^{\frac{1}{2}D_n} a\cdot a\big)\cdot\big(e^{\frac{1}{2}D_n} a\cdot a\big)+\big(D_t^2 e^{\frac{1}{2}D_n} a\cdot a\big)\cdot\big(D_t e^{\frac{1}{2}D_n} a\cdot a\big)\big]$$
$$=\tfrac{1}{2}D_t\big[\big(D_t^2 e^{\frac{1}{2}D_n} a\cdot a\big)\cdot a^2+\big(e^{D_n} a\cdot a\big)\cdot\big(D_t^2 a\cdot a\big)\big]$$

and

$$\sinh\big(\tfrac{1}{2}D_n\big)\big[\big(D_z D_t e^{\frac{1}{2}D_n} a\cdot a\big)\cdot\big(e^{\frac{1}{2}D_n} a\cdot a\big)+\big(D_z e^{\frac{1}{2}D_n} a\cdot a\big)\cdot\big(D_t e^{\frac{1}{2}D_n} a\cdot a\big)\big]$$
$$=D_t\cosh\big(\tfrac{1}{2}D_n\big)\big(D_z e^{\frac{1}{2}D_n} a\cdot a\big)\cdot\big(e^{\frac{1}{2}D_n} a\cdot a\big)$$

we can obtain the following multilinear equation from (4) and (5):

$$\begin{aligned}
&8\sinh\big(\tfrac{1}{2}D_n\big)\big(D_t^2 e^{\frac{1}{2}D_n} f(n)\cdot f(n)\big)\cdot\big(D_t e^{\frac{1}{2}D_n} f(n)\cdot f(n)\big)\\
&\quad+2\sinh\big(\tfrac{1}{2}D_n\big)\big(D_t e^{\frac{1}{2}D_n} f(n)\cdot f(n)\big)\cdot\big(e^{\frac{1}{2}D_n} f(n)\cdot f(n)\big)\\
&\quad+3\alpha\sinh\big(\tfrac{1}{2}D_n\big)\big(D_t e^{\frac{3}{2}D_n} f(n)\cdot f(n)\big)\cdot\big(e^{\frac{1}{2}D_n} f(n)\cdot f(n)\big)\\
&\quad-3\alpha\sinh\big(\tfrac{1}{2}D_n\big)\big[\big(e^{\frac{3}{2}D_n}-e^{\frac{1}{2}D_n}\big)f(n)\cdot f(n)\big]\cdot\big(D_t e^{\frac{1}{2}D_n} f(n)\cdot f(n)\big)\\
&=D_t\big[\big(D_t^2 e^{D_n} f(n)\cdot f(n)\big)\cdot f^2(n)+\big(e^{D_n} f(n)\cdot f(n)\big)\cdot\big(D_t^2 f(n)\cdot f(n)\big)\big]\\
&\quad+6D_t\cosh\big(\tfrac{1}{2}D_n\big)\big(D_t^2 e^{\frac{1}{2}D_n} f(n)\cdot f(n)\big)\cdot\big(e^{\frac{1}{2}D_n} f(n)\cdot f(n)\big)\\
&\quad-3\alpha D_t\cosh\big(\tfrac{1}{2}D_n\big)\big(e^{\frac{3}{2}D_n} f(n)\cdot f(n)\big)\cdot\big(e^{\frac{1}{2}D_n} f(n)\cdot f(n)\big).
\end{aligned}\tag{6}$$

Set

$$u(n)=(\ln f(n))_t \qquad w(n)=\frac{f(n+2)f(n-1)}{f(n+1)f(n)}.$$

Then we can deduce the following system:

$$\begin{aligned}
&u_{tt}(n+1)+u_{tt}(n)+u_{tt}(n-1)-3u(n)(u_t(n+1)+u_t(n-1))+3u(n+1)u_t(n+1)\\
&\quad+3u(n-1)u_t(n-1)-\tfrac{1}{4}u(n+1)-\tfrac{1}{4}u(n-1)\\
&\quad+\tfrac{1}{2}u(n)+[u(n+1)-2u(n)+u(n-1)][(u(n+1)-u(n-1))^2\\
&\quad-(u(n+1)-u(n))(u(n)-u(n-1))]-\tfrac{3}{4}\alpha(u(n+2)-u(n))w(n)\\
&\quad-\tfrac{3}{4}\alpha(u(n-2)-u(n))w(n-1)+\tfrac{3}{8}\alpha(u(n+1)+u(n-1)-2u(n))=0
\end{aligned}\tag{7}$$

$$w_t(n)=w(n)(u(n+2)-u(n+1)-u(n)+u(n-1)).\tag{8}$$

Obviously, when $\alpha=0$, equation (7) becomes (1). The system (7) and (8) can be rewritten as the following equation for a single field:

$$\begin{aligned}
&U_{ttt}(n+1)+U_{ttt}(n)+U_{ttt}(n-1)-3U_t(n)(U_{tt}(n+1)+U_{tt}(n-1))\\
&\quad+3U_t(n+1)U_{tt}(n+1)+3U_t(n-1)U_{tt}(n-1)-\tfrac{1}{4}U_t(n+1)-\tfrac{1}{4}U_t(n-1)\\
&\quad+\tfrac{1}{2}U_t(n)+[U_t(n+1)-2U_t(n)+U_t(n-1)][(U_t(n+1)-U_t(n-1))^2\\
&\quad-(U_t(n+1)-U_t(n))(U_t(n)-U_t(n-1))]\\
&\quad-\tfrac{3}{4}\alpha(U_t(n+2)-U_t(n))e^{U(n+2)-U(n+1)-U(n)+U(n-1)}\\
&\quad-\tfrac{3}{4}\alpha(U_t(n-2)-U_t(n))e^{U(n+1)-U(n)-U(n-1)+U(n-2)}\\
&\quad+\tfrac{3}{8}\alpha(U_t(n+1)+U_t(n-1)-2U_t(n))=0
\end{aligned}\tag{9}$$

where $u(n)=U_t(n)$.

Concerning (4) and (5), we have the following result.

Proposition. A Bäcklund transformation for equations (4) and (5) is

$$\left(D_t + \lambda^{-1}\mathrm{e}^{-D_n} - \tfrac{1}{4}\alpha\lambda\mathrm{e}^{D_n} + \mu\right) f(n)\cdot g(n) = 0 \tag{10}$$

$$\left(D_z - \lambda^{-1}D_t\mathrm{e}^{-D_n} - \tfrac{1}{4}\alpha\lambda D_t\mathrm{e}^{D_n} - \lambda^{-1}\mu\mathrm{e}^{-D_n} - \tfrac{1}{4}\alpha\lambda\mu\mathrm{e}^{D_n} - \omega\right) f(n)\cdot g(n) = 0 \tag{11}$$

$$\begin{aligned}&\big(D_t^3 + 3\mu D_t^2 - 3\lambda^{-1}D_t^2\mathrm{e}^{-D_n} - 6\lambda^{-1}\mu D_t\mathrm{e}^{-D_n} - 6\lambda^{-1}\mu^2\mathrm{e}^{-D_n} - D_t + \tfrac{3}{4}\alpha\lambda D_t^2\mathrm{e}^{D_n}\\ &\quad + \tfrac{3}{2}\alpha\lambda\mu D_t\mathrm{e}^{D_n} + \tfrac{3}{2}\alpha\lambda\mu^2\mathrm{e}^{D_n} + \tfrac{3}{2}\alpha D_t + \gamma\big) f(n)\cdot g(n) = 0\end{aligned} \tag{12}$$

where λ, μ, γ and ω are arbitrary constants.

Proof. Let $f(n)$ be a solution of equations (4) and (5). If we can show that equations (10)–(12) guarantee that the following two relations:

$$P_1 \equiv \left(D_z\mathrm{e}^{\frac{1}{2}D_n} - D_t^2\mathrm{e}^{\frac{1}{2}D_n} + \tfrac{1}{2}\alpha\left(\mathrm{e}^{\frac{3}{2}D_n} - \mathrm{e}^{\frac{1}{2}D_n}\right)\right) g(n)\cdot g(n) = 0$$

$$P_2 \equiv \left(D_t^3\mathrm{e}^{\frac{1}{2}D_n} + 3D_tD_z\mathrm{e}^{\frac{1}{2}D_n} - D_t\mathrm{e}^{\frac{1}{2}D_n} - \tfrac{3}{2}\alpha D_t\mathrm{e}^{\frac{3}{2}D_n}\right) g(n)\cdot g(n) = 0$$

hold, then equations (10)–(12) form a Bäcklund transformation. In fact, $P_1 = 0$ can be proved similarly to those in [7], while $P_2 = 0$ can be shown similarly as in [1]. The details of the proof are omitted. □

In what follows, we want to show that (7) and (8) are integrable in the sense of a Lax representation. In order to derive a Lax pair for (7) and (8), we set in (10)–(12) that

$$u(n) = (\ln g(n))_t,\ w(n) = \frac{g(n+2)g(n-1)}{g(n+1)g(n)} \qquad f(n) = g(n+1)\psi_n \tag{13}$$

from which it implies that

$$w_t(n) = w(n)(u(n+2) - u(n+1) - u(n) + u(n-1)).$$

Substituting (13) into (10) and (12), we obtain

$$\psi_{n,t} + (u(n+1) - u(n) + \mu)\psi_n + \lambda^{-1}\psi_{n-1} - \tfrac{1}{4}\alpha\lambda w(n)\psi_{n+1} = 0 \tag{14}$$

$$\begin{aligned}&\psi_{n,ttt} + 3(u(n+1) - u(n) + \mu)\psi_{n,tt} - 3\lambda^{-1}\psi_{n-1,tt} + \tfrac{3}{4}\alpha\lambda w(n)\psi_{n+1,tt}\\
&\quad +3\big[(u(n+1) + u(n))_t + (u(n+1) - u(n))^2 + 2\mu(u(n+1)\\
&\quad -u(n)) - \tfrac{1}{3} + \tfrac{1}{2}\alpha\big]\psi_{n,t}\\
&\quad +6\lambda^{-1}(u(n+1) - u(n) - \mu)\psi_{n-1,t}\\
&\quad +\tfrac{3}{2}\alpha\lambda w(n)\psi_{n+1,t}(u(n+2) - u(n-1) + \mu)\\
&\quad +\big[(u(n+1) - u(n))_{tt} + 3(u(n+1) - u(n))(u_t(n+1) + u_t(n))\\
&\quad +(u(n+1) - u(n))^3 + 3\mu(u_t(n+1) + u_t(n) + (u(n+1) - u(n))^2)\\
&\quad +\left(\tfrac{3}{2}\alpha - 1\right)(u(n+1) - u(n)) + \gamma\big]\psi_n\\
&\quad -3\lambda^{-1}[(u(n+1) + u(n))_t + (u(n+1) - u(n))^2\\
&\quad -2\mu(u(n+1) - u(n)) + 2\mu^2]\psi_{n-1} + \tfrac{3}{4}\alpha\lambda w(n)\psi_{n+1}[u_t(n+2)\\
&\quad +u_t(n-1) + (u(n+2) - u(n-1))^2 + 2(u(n+2) - u(n-1)) + 2\mu] = 0.\end{aligned} \tag{15}$$

From (14) and (15), we have

$$-4\psi_{n-3} + 4\lambda(u(n-2) - u(n+1))\psi_{n-2} + \tfrac{1}{2}\lambda^2[2 - 3\alpha - 8u^2(n-1)$$
$$-8u^2(n) + 8u(n)u(n+1) - 8u^2(n+1) + 8u(n-1)(u(n) + u(n+1))$$
$$+2\alpha w(n-2) + 2\alpha w(n-1) + 2\alpha w(n) - 8u_t(n-1)$$
$$-8u_t(n) - 8u_t(n+1)]\psi_{n-1}$$
$$+\tfrac{1}{2}\lambda^3[2\gamma + 2\mu - 3\alpha\mu + 4\mu^3 - 2\alpha u(n-2)w(n-1)$$
$$+2\alpha u(n+1)w(n-1) + 2\alpha u(n-1)w(n) - 2\alpha u(n+2)w(n)]\psi_n$$
$$-\tfrac{1}{8}\alpha\lambda^4 w(n)[2 - 3\alpha - 8u^2(n-1) - 8u^2(n) + 8u(n)u(n+1) - 8u^2(n+1)$$
$$+8u(n-1)(u(n) + u(n+1)) + 2\alpha w(n-1) + 2\alpha w(n)$$
$$+2\alpha w(n+1) - 8u_t(n-1) - 8u_t(n) - 8u_t(n+1)]\psi_{n+1}$$
$$+\tfrac{1}{4}\alpha^2\lambda^5 w(n)w(n+1)(-u(n-1) + u(n+2))\psi_{n+2}$$
$$+\tfrac{1}{16}\alpha^3\lambda^6 w(n)w(n+1)w(n+2)\psi_{n+3} = 0. \qquad (16)$$

Equations (14) and (16) can be rewritten as

$$\psi_{n,t} + (U_t(n+1) - U_t(n) + \mu)\psi_n + \lambda^{-1}\psi_{n-1} - \tfrac{1}{4}\alpha\lambda e^{U(n+2)-U(n+1)-U(n)+U(n-1)}\psi_{n+1} = 0 \qquad (17)$$

$$-4\psi_{n-3} + 4\lambda(U_t(n-2) - U_t(n+1))\psi_{n-2} + \tfrac{1}{2}\lambda^2[2 - 3\alpha - 8U_t^2(n-1)$$
$$-8U_t^2(n) + 8U_t(n)U_t(n+1) - 8U_t^2(n+1) + 8U_t(n-1)(U_t(n) + U_t(n+1))$$
$$+2\alpha e^{U(n)-U(n-1)-U(n-2)+U(n-3)} + 2\alpha e^{U(n+1)-U(n)-U(n-1)+U(n-2)}$$
$$+2\alpha e^{U(n+2)-U(n+1)-U(n)+U(n-1)} - 8U_{tt}(n-1) - 8U_{tt}(n) - 8U_{tt}(n+1)]\psi_{n-1}$$
$$+\tfrac{1}{2}\lambda^3[2\gamma + 2\mu - 3\alpha\mu + 4\mu^3 - 2\alpha U_t(n-2)e^{U(n+1)-U(n)-U(n-1)+U(n-2)}$$
$$+2\alpha U_t(n+1)e^{U(n+1)-U(n)-U(n-1)+U(n-2)}$$
$$+2\alpha U_t(n-1)e^{U(n+2)-U(n+1)-U(n)+U(n-1)}$$
$$-2\alpha U_t(n+2)e^{U(n+2)-U(n+1)-U(n)+U(n-1)}]\psi_n$$
$$-\tfrac{1}{8}\alpha\lambda^4 e^{U(n+2)-U(n+1)-U(n)+U(n-1)}[2 - 3\alpha - 8U_t^2(n-1) - 8U_t^2(n)$$
$$+8U_t(n)U_t(n+1) - 8U_t^2(n+1) + 8U_t(n-1)(U_t(n) + U_t(n+1))$$
$$+2\alpha e^{U(n+1)-U(n)-U(n-1)+U(n-2)} + 2\alpha e^{U(n+2)-U(n+1)-U(n)+U(n-1)}$$
$$+2\alpha e^{U(n+3)-U(n+2)-U(n+1)+U(n)} - 8U_{tt}(n-1) - 8U_{tt}(n) - 8U_{tt}(n+1)]\psi_{n+1}$$
$$+\tfrac{1}{4}\alpha^2\lambda^5 e^{U(n+3)-2U(n+1)+U(n-1)}(-U_t(n-1) + U_t(n+2))\psi_{n+2}$$
$$+\tfrac{1}{16}\alpha^3\lambda^6 e^{U(n+4)-U(n+2)-U(n+1)+U(n-1)}\psi_{n+3} = 0. \qquad (18)$$

Thus, equations (17) and (18) or (14) and (16) constitute a Lax pair for (9) or (7) and (8), respectively, which can be checked by using *Mathematica* [8] directly.

Finally, we consider a continuum approximation of equation (9). In order to do so, we rewrite (9) as

$$u_{tt}(n+1) + u_{tt}(n) + u_{tt}(n-1) - 3u(n)(u_t(n+1) + u_t(n-1))$$
$$+3u(n+1)u_t(n+1) + 3u(n-1)u_t(n-1) - \tfrac{1}{4}u(n+1) - \tfrac{1}{4}u(n-1)$$
$$+\tfrac{1}{2}u(n) + [u(n+1) - 2u(n) + u(n-1)][(u(n+1) - u(n-1))^2$$
$$-(u(n+1) - u(n))(u(n) - u(n-1))]$$

$$-\tfrac{3}{4}\alpha(u(n+2)-u(n))\mathrm{e}^{\int^{t}(u(n+2)-u(n+1)-u(n)+u(n-1))\,\mathrm{d}t'}$$
$$-\tfrac{3}{4}\alpha(u(n-2)-u(n))\mathrm{e}^{\int^{t}(u(n+1)-u(n)-u(n-1)+u(n-2))\,\mathrm{d}t'}$$
$$+\tfrac{3}{8}\alpha(u(n+1)+u(n-1)-2u(n))=0. \tag{19}$$

We now take the dependent variable $u(n)=A+\epsilon^2 v(n)$ with A a constant and expand $v(n+1)$ as

$$\begin{aligned} v(n+1) &= v+\epsilon\frac{\partial}{\partial x}v+\frac{\epsilon^2}{2}\frac{\partial^2}{\partial x^2}v+\frac{\epsilon^3}{6}\frac{\partial^3}{\partial x^3}v+\frac{\epsilon^4}{24}\frac{\partial^4}{\partial x^4}v+\cdots \\ v_t(n+1) &= \epsilon^2\left[v_t+\epsilon\frac{\partial^2}{\partial x\partial t}v+\frac{\epsilon^2}{2}\frac{\partial^3}{\partial x^2\partial t}v+\frac{\epsilon^3}{6}\frac{\partial^4}{\partial x^3\partial t}v+\cdots\right] \end{aligned} \tag{20}$$

where we have made the transformations $\partial/\partial t\to\epsilon^2\partial/\partial t$, $\partial/\partial n\to\epsilon\partial/\partial x$. Substituting (20), etc into (19) and neglecting higher-order terms of ϵ, we obtain

$$\begin{aligned} &v_{tt}+v_t v_{xx}+2v_x v_{tx}+\left(A-\tfrac{1}{12}+\tfrac{1}{8}\alpha\right)\left(\epsilon^{-2}v_{xx}+\tfrac{1}{12}v_{xxxx}\right) \\ &\quad +v_x^2 v_{xx}-\alpha\epsilon^{-2}\left(v_{xx}+v_x\int^t v_{xxx}\,\mathrm{d}t'\right)\mathrm{e}^{2\int^t v_{xx}\,\mathrm{d}t'} \\ &\quad -\alpha\left(\mathrm{e}^{2\int^t v_{xx}\,\mathrm{d}t'}\right)\left[\tfrac{1}{3}v_{xxxx}+\tfrac{2}{3}v_{xxx}\int^t v_{xxx}\,\mathrm{d}t'\right. \\ &\quad +\tfrac{2}{3}v_{xx}\int^t v_{xxxx}\,\mathrm{d}t'+\tfrac{1}{2}v_{xx}\left(\int^t v_{xxx}\,\mathrm{d}t'\right)^2+\tfrac{4}{15}v_x\int^t v_{xxxxx}\,\mathrm{d}t' \\ &\quad \left.+\tfrac{2}{3}v_x\left(\int^t v_{xxx}\,\mathrm{d}t'\right)\left(\int^t v_{xxxx}\,\mathrm{d}t'\right)+\tfrac{1}{6}v_x\left(\int^t v_{xxx}\,\mathrm{d}t'\right)^3\right]=0. \end{aligned} \tag{21}$$

In particular, if we choose $\alpha=0$ and $A=\frac{365}{6}$, equation (21) becomes the Kaup equation [9]

$$w_{tt}+\tfrac{27}{\epsilon^2}w_{xx}+w_{xxxx}+\tfrac{1}{2}\left(w_x^2\right)_t+\left(w_x w_t+\tfrac{1}{2}w_x^3\right)_x=0 \tag{22}$$

under the transformation

$$x\longrightarrow\tfrac{2}{3}x \qquad t\longrightarrow t \qquad w=\tfrac{9}{4}v.$$

To summarize, in this short paper, an integrable differential-difference system is proposed which is an extended form of the integrable differential-difference equation found in [1]. We have shown that this extended differential-difference system (7) and (8) is integrable in the sense of having a Bäcklund transformation and a Lax pair. The continuum approximation of this extended equation (19) has also been considered. Besides, it would be of interest to extend the differential-difference equation (19) to a corresponding difference-difference equation, which is equivalent to considering a difference-difference version of bilinear form (4) and (5). As a discussion, for the bilinear equations (2) and (3) we conjecture that their difference-difference version is

$$\left(z_1\mathrm{e}^{D_1}+z_2\mathrm{e}^{D_2}+z_3\mathrm{e}^{D_3}\right)f\cdot f=0 \tag{23}$$

$$\left(z_1\mathrm{e}^{D_1-D_3}+z_2\mathrm{e}^{D_2-D_3}+z_3\mathrm{e}^{2D_3}\right)f\cdot f=0 \tag{24}$$

although we have not confirmed it, where $D_i=\epsilon_i D_x+\delta_i D_y+\theta_i D_t$ and z_i,ϵ_i,δ_i and θ_i are arbitrary parameters. Concerning (23) and (24), we have found a bilinear Bäcklund transformation. Since (4) and (5) are the extended forms of (2) and (3), it would also be natural for us to conjecture that a difference-difference version for (4) and (5) might be some generalized equations of (23) and (24), one of which is the following discrete BKP equation:

$$\left(z_1\mathrm{e}^{D_1}+z_2\mathrm{e}^{D_2}+z_3\mathrm{e}^{D_3}+z_4\mathrm{e}^{D_1+D_2+D_3}\right)f\cdot f=0. \tag{25}$$

We hope these problems will be solved.

Acknowledgments

The authors would like to express their thanks to the referees for their valuable comments and suggestion. This work was supported by the National Natural Science Foundation of China (project no 19871082), the China State Major Key Project for Basic Researches, and the Chinese Academy of Sciences.

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